

MODULES OVER ÉTALE GROUPOID ALGEBRAS AS SHEAVES

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ABSTRACT. The author has previously associated to each commutative ring with unit \mathbb{k} and étale groupoid \mathcal{G} with locally compact, Hausdorff, totally disconnected unit space a \mathbb{k} -algebra $\mathbb{k}\mathcal{G}$. The algebra $\mathbb{k}\mathcal{G}$ need not be unital, but it always has local units. The class of groupoid algebras includes group algebras, inverse semigroup algebras and Leavitt path algebras. In this paper we show that the category of unitary $\mathbb{k}\mathcal{G}$ -modules is equivalent to the category of sheaves of \mathbb{k} -modules over \mathcal{G} . As a consequence we obtain a new proof of a recent result that Morita equivalent groupoids have Morita equivalent algebras.

1. INTRODUCTION

In an effort to obtain a uniform theory for group algebras, inverse semigroup algebras and Leavitt path algebras [2], the author [25] associated to each commutative ring with unit \mathbb{k} and étale groupoid \mathcal{G} with locally compact, totally disconnected unit space a \mathbb{k} -algebra $\mathbb{k}\mathcal{G}$ (see also [6]). These algebras are discrete versions of groupoid C^* -algebras [20, 22] and a number of analogues of results from the operator theoretic setting have been obtained in this context. In particular, Cuntz-Krieger uniqueness theorems [4, 5], characterizations of simplicity [4, 5] and the connection of groupoid Morita equivalence to Morita equivalence of algebras [7] have been proven for groupoid algebras under the Hausdorff assumption.

In this paper, we prove a discrete analogue of Renault's disintegration theorem [23], which roughly states that representations of groupoid C^* -algebras are obtained by integrating representations of the groupoid. A representation of a groupoid consists of a field of Hilbert spaces over the unit space with an action of the groupoid by unitary transformations on the fibers [20, 22].

Here we prove that the category of unitary $\mathbb{k}\mathcal{G}$ -modules is equivalent to the category of sheaves of \mathbb{k} -modules over \mathcal{G} . This simultaneously generalizes

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the following two well-known facts: if X is a locally compact, totally disconnected space and $C_c(X, \mathbb{k})$ is the ring of locally constant functions $X \rightarrow \mathbb{k}$ with compact support, then the category of unitary $C_c(X, \mathbb{k})$ -modules is equivalent to the category of sheaves on X (cf. [21]); and if \mathcal{G} is a discrete groupoid, then the category of unitary $\mathbb{k}\mathcal{G}$ -modules is equivalent to the category of functors from \mathcal{G} to the category of \mathbb{k} -modules (cf. [15]). In the context of étale Lie groupoids and convolution algebras of smooth functions, analogous results can be found in [13]. However, the techniques in the totally disconnected case setting are quite different.

As a consequence of our results, we obtain a new proof that Morita equivalent groupoids have Morita equivalent groupoid algebras, which the author feels is more conceptual than the one in [7] (since it works with module categories rather than Morita contexts), and at the same time does not require the Hausdorff hypothesis.

We hope that this geometric realization of the module category will prove useful in the study of simple modules, primitive ideals and in other contexts analogous to those in which Renault's disintegration theorem is used in operator theory.

2. ÉTALE GROUPOIDS

In this paper, a topological space will be called compact if it is Hausdorff and satisfies the property that every open cover has a finite subcover.

2.1. Topological groupoids. A topological groupoid \mathcal{G} is a groupoid (i.e., a small category each of whose morphisms is an isomorphism) whose unit space $\mathcal{G}^{(0)}$ and arrow space $\mathcal{G}^{(1)}$ are topological spaces and whose domain map \mathbf{d} , range map \mathbf{r} , multiplication map, inversion map and unit map $u: \mathcal{G}^{(0)} \rightarrow \mathcal{G}^{(1)}$ are all continuous. Since u is a homeomorphism with its image, we identify elements of $\mathcal{G}^{(0)}$ with the corresponding identity arrows and view $\mathcal{G}^{(0)}$ as a subspace of $\mathcal{G}^{(1)}$ with the subspace topology. We write $\mathcal{G}^{(2)}$ for the space of composable arrows (g, h) with $\mathbf{d}(g) = \mathbf{r}(h)$.

A topological groupoid \mathcal{G} is *étale* if \mathbf{d} is a local homeomorphism. This implies that \mathbf{r} and the multiplication map are local homeomorphisms and that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}^{(1)}$ [24]. Note that the fibers of \mathbf{d} and \mathbf{r} are discrete in the induced topology. A *local bisection* of \mathcal{G} is an open subset $U \subseteq \mathcal{G}^{(1)}$ such that $\mathbf{d}|_U$ and $\mathbf{r}|_U$ are homeomorphisms to their images. The set of local bisections of \mathcal{G} , denoted $\text{Bis}(\mathcal{G})$, is a basis for the topology on $\mathcal{G}^{(1)}$ [9, 20, 24]. If U, V are local bisections, then

$$UV = \{uv \mid u \in U, v \in V\}$$

$$U^{-1} = \{u^{-1} \mid u \in U\}$$

are local bisections. In fact, $\text{Bis}(\mathcal{G})$ is an inverse semigroup [14].

An étale groupoid is said to be *ample* [20] if $\mathcal{G}^{(0)}$ is Hausdorff and has a basis of compact open sets. In this case $\mathcal{G}^{(1)}$ is locally Hausdorff but

need not be Hausdorff. Let $\text{Bis}_c(\mathcal{G})$ denote the set of compact open local bisections of \mathcal{G} . Then $\text{Bis}_c(\mathcal{G})$ is an inverse subsemigroup of $\text{Bis}(\mathcal{G})$ and is a basis for the topology of $\mathcal{G}^{(1)}$ [20].

2.2. \mathcal{G} -sheaves and Morita equivalence. Let \mathcal{G} be an étale groupoid. References for this section are [8, 11, 12, 16–19]. A (*right*) \mathcal{G} -space consists of a space E , a continuous map $p: E \rightarrow \mathcal{G}^{(0)}$ and an action map $E \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \rightarrow E$ (where the fiber product is with respect to p and \mathbf{r}), denoted $(x, g) \mapsto xg$ satisfying the following axioms:

- $ep(e) = e$ for all $e \in E$;
- $p(eg) = \mathbf{d}(g)$ whenever $p(e) = \mathbf{r}(g)$;
- $(eg)h = e(gh)$ whenever $p(e) = \mathbf{r}(g)$ and $\mathbf{d}(g) = \mathbf{r}(h)$.

Left \mathcal{G} -spaces are defined dually.

A \mathcal{G} -space (E, p) is said to be *principal* if the natural map

$$E \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} \rightarrow E \times_{\mathcal{G}^{(0)}} E$$

given by $(e, g) \mapsto (eg, e)$ is a homeomorphism. A *morphism* $(E, p) \rightarrow (F, q)$ of \mathcal{G} -spaces is a continuous map $\varphi: E \rightarrow F$ such that

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & \searrow p & \swarrow q \\ & \mathcal{G}^{(0)} & \end{array}$$

commutes and $\varphi(eg) = \varphi(e)g$ whenever $p(e) = \mathbf{r}(g)$.

Morita equivalence plays an important role in groupoid theory. There are a number of different, but equivalent, formulations of the notion. See [8, 11, 12, 16–19] for details. Two topological groupoids \mathcal{G} and \mathcal{H} are said to be *Morita equivalent* if there is a topological space E with the structure of a principal left \mathcal{G} -space (E, p) and a principal right \mathcal{H} -space (E, q) such that p, q are open surjections and the actions commute, meaning $p(eh) = p(e)$, $q(ge) = q(e)$ and $(ge)h = g(eh)$ whenever $g \in \mathcal{G}^{(1)}$, $h \in \mathcal{H}^{(1)}$ and $\mathbf{d}(g) = p(e)$, $\mathbf{r}(h) = q(e)$.

A continuous functor $f: \mathcal{G} \rightarrow \mathcal{H}$ of étale groupoids is called an *essential equivalence* if $\mathbf{d}\pi_2: \mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(0)}$ is an open surjection (where the fiber product is over f and \mathbf{r}) and the square

$$\begin{array}{ccc} \mathcal{G}^{(1)} & \xrightarrow{f} & \mathcal{H}^{(1)} \\ (d, r) \downarrow & & \downarrow (d, r) \\ \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} & \xrightarrow{f \times f} & \mathcal{H}^{(0)} \times \mathcal{H}^{(0)} \end{array}$$

is a pullback. The first condition corresponds to being essentially surjective and the second to being fully faithful in the discrete context. Étale groupoids \mathcal{G} and \mathcal{H} are Morita equivalent if and only if there is an étale groupoid \mathcal{K} and essential equivalences $f: \mathcal{K} \rightarrow \mathcal{G}$ and $f': \mathcal{K} \rightarrow \mathcal{H}$.

If \mathcal{G} is an étale groupoid, then a \mathcal{G} -sheaf consists of a \mathcal{G} -space (E, p) such that $p: E \rightarrow \mathcal{G}^{(0)}$ is a local homeomorphism (the tradition to use right actions is standard in topos theory). The fiber $p^{-1}(x)$ of E over x is denoted E_x and is called the *stalk* of E at x . A morphism of \mathcal{G} -sheaves is just a morphism of \mathcal{G} -spaces; note, however, that the corresponding map of total spaces must necessarily be a local homeomorphism. The category $\mathcal{B}\mathcal{G}$ of all \mathcal{G} -sheaves is called the *classifying topos* of \mathcal{G} [11, 12].

If A is a set, then the constant \mathcal{G} -sheaf $\Delta(A)$ is $(A \times \mathcal{G}^{(0)}, \pi_2)$ with action $(a, \mathbf{r}(g))g = (a, \mathbf{d}(g))$ (where A is endowed with the discrete topology). As a sheaf over $\mathcal{G}^{(0)}$, note that $\Delta(A)$ is nothing more than the sheaf of locally constant A -valued functions on $\mathcal{G}^{(0)}$. (Recall that a locally constant function from a topological space X to a set A is just a continuous map $X \rightarrow A$ where A is endowed with the discrete topology.) The functor $\Delta: \mathbf{Set} \rightarrow \mathcal{B}\mathcal{G}$ is exact and hence sends rings to internal rings of $\mathcal{B}\mathcal{G}$. If \mathbb{k} is a commutative ring with unit, then a \mathcal{G} -sheaf of \mathbb{k} -modules is by definition an internal $\Delta(\mathbb{k})$ -module in $\mathcal{B}\mathcal{G}$. Explicitly, this amounts to a \mathcal{G} -sheaf (E, p) together a \mathbb{k} -module structure on each stalk E_x such that:

- the zero section, denoted 0 , sending $x \in \mathcal{G}^{(0)}$ to the zero of E_x is continuous;
- addition $E \times_{\mathcal{G}^{(0)}} E \rightarrow E$ is continuous;
- scalar multiplication $K \times E \rightarrow E$ is continuous;
- for each $g \in \mathcal{G}^{(1)}$, the map $R_g: E_{\mathbf{r}(g)} \rightarrow E_{\mathbf{d}(g)}$ given by $R_g(e) = eg$ is \mathbb{k} -linear;

where \mathbb{k} has the discrete topology in the third item. Note that the first three conditions are equivalent to (E, p) being a sheaf of \mathbb{k} -modules over $\mathcal{G}^{(0)}$.

Internal $\Delta(\mathbb{k})$ -module homomorphisms are just \mathcal{G} -sheaf morphisms

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & \searrow p \quad \swarrow q & \\ & \mathcal{G}^{(0)} & \end{array}$$

which restrict to \mathbb{k} -module homomorphisms on the stalks. The category of \mathcal{G} -sheaves of \mathbb{k} -modules will be denoted $\mathcal{B}_{\mathbb{k}}\mathcal{G}$.

It follows from standard topos theory that if $\mathcal{B}\mathcal{G}$ is equivalent to $\mathcal{B}\mathcal{H}$, then the equivalence commutes with the constant functor (up to isomorphism) and hence yields an equivalence of categories $\mathcal{B}_{\mathbb{k}}\mathcal{G}$ and $\mathcal{B}_{\mathbb{k}}\mathcal{H}$. Indeed, Δ is left adjoint to the hom functor out of the terminal object and equivalences preserve terminal objects.

Moerdijk proved that if \mathcal{G} and \mathcal{H} are étale groupoids, then $\mathcal{B}\mathcal{G}$ is equivalent to $\mathcal{B}\mathcal{H}$ if and only if \mathcal{G} and \mathcal{H} are Morita equivalent groupoids [8, 11, 12, 16–18]. Hence Morita equivalent groupoids have equivalent categories of sheaves of \mathbb{k} -modules for any base ring \mathbb{k} .

2.3. Groupoid algebras. Let \mathcal{G} be an ample groupoid and \mathbb{k} a commutative ring with unit. Define $\mathbb{k}\mathcal{G}$ to be the \mathbb{k} -submodule of $\mathbb{k}^{\mathcal{G}^{(1)}}$ spanned by the characteristic functions χ_U with $U \in \text{Bis}_c(\mathcal{G})$. If $\mathcal{G}^{(1)}$ is Hausdorff, then $\mathbb{k}\mathcal{G}$ consists precisely of the locally constant functions $\mathcal{G}^{(1)} \rightarrow \mathbb{k}$ with compact support; otherwise, it is the \mathbb{k} -submodule spanned by those functions $f: \mathcal{G}^{(1)} \rightarrow \mathbb{k}$ that vanish outside some Hausdorff open subset U with $f|_U$ locally constant with compact support. See [6, 25, 26] for details.

The convolution product on $\mathbb{k}\mathcal{G}$, defined by

$$f_1 * f_2(g) = \sum_{\mathbf{d}(h)=\mathbf{d}(g)} f_1(gh^{-1})f_2(h),$$

turns $\mathbb{k}\mathcal{G}$ into a \mathbb{k} -algebra. Note that the sum is finite because the fibers of \mathbf{d} are closed and discrete, and f_1, f_2 are linear combinations of functions with compact support. We often just write for convenience $f_1 f_2$ instead of $f_1 * f_2$. One has that $\chi_U \chi_V = \chi_{UV}$ for $U, V \in \text{Bis}_c(\mathcal{G})$; see [25]. If $\mathcal{G}^{(1)} = \mathcal{G}^{(0)}$, then the convolution product is just pointwise multiplication and so $\mathbb{k}\mathcal{G}$ is just the usual ring of locally constant functions $\mathcal{G}^{(0)} \rightarrow \mathbb{k}$ with compact support.

The ring $\mathbb{k}\mathcal{G}$ is unital if and only if $\mathcal{G}^{(0)}$ is compact [25]. However, it is very close to being unital in the following sense. A ring R is said to have *local units* if it is a direct limit of unital rings in the category of not necessarily unital rings (that is, the homomorphisms in the directed system do not have to preserve the identities). Equivalently, R has local units if, for any finite subset r_1, \dots, r_n of R , there is an idempotent $e \in R$ with $er_i = r_i = r_i e$ for $i = 1, \dots, n$ [1, 3]. Denote by $E(R)$ the set of idempotents of R .

Proposition 2.1. *Let \mathcal{G} be an ample groupoid and \mathbb{k} a commutative ring with units. Let \mathcal{B} denote the generalized boolean algebra of compact open subsets of $\mathcal{G}^{(0)}$. If $U \in \mathcal{B}$, let $\mathcal{G}_U = (U, \mathbf{d}^{-1}(U) \cap \mathbf{r}^{-1}(U))$.*

- (1) \mathcal{B} is directed.
- (2) If $U \in \mathcal{B}$, then \mathcal{G}_U is an open ample subgroupoid of \mathcal{G} .
- (3) If $U \in \mathcal{B}$, then $\chi_U \cdot \mathbb{k}\mathcal{G} \cdot \chi_U \cong \mathbb{k}\mathcal{G}_U$.
- (4) $\mathbb{k}\mathcal{G} = \bigcup_{U \in \mathcal{B}} \chi_U \cdot \mathbb{k}\mathcal{G} \cdot \chi_U = \varinjlim_{U \in \mathcal{B}} \mathbb{k}\mathcal{G}_U$.

In particular, $\mathbb{k}\mathcal{G}$ has local units.

Proof. Clearly, \mathcal{B} is directed since the union of two elements is their join. Also \mathcal{G}_U is an open ample subgroupoid of \mathcal{G} . It follows that $\mathbb{k}\mathcal{G}_U$ can be identified with a subalgebra of $\mathbb{k}\mathcal{G}$ by extending functions on $\mathcal{G}_U^{(1)}$ to be 0 outside of $\mathcal{G}_U^{(1)}$. Since χ_U is the identity of $\mathbb{k}\mathcal{G}_U$ (cf. [25]), $\mathbb{k}\mathcal{G}_U$ is a unital subring of $\chi_U \cdot \mathbb{k}\mathcal{G} \cdot \chi_U$. But if $f \in \mathbb{k}\mathcal{G}$ and $g \notin \mathcal{G}_U^{(1)}$, then $(\chi_U \cdot f \cdot \chi_U)(g) = 0$. Thus $\chi_U \cdot \mathbb{k}\mathcal{G} \cdot \chi_U = \mathbb{k}\mathcal{G}_U$.

Let $R = \bigcup_{U \in \mathcal{B}} \chi_U \cdot \mathbb{k}\mathcal{G}_U \cdot \chi_U$. Then R is a \mathbb{k} -subalgebra of $\mathbb{k}\mathcal{G}$ because \mathcal{B} is directed. To show that R is the whole ring, we just need to show it contains the spanning set χ_U with $U \in \text{Bis}_c(\mathcal{G})$. Put $V = U^{-1}U \cup UU^{-1}$. Then $V \in \mathcal{B}$ and $\chi_V \cdot \chi_U \cdot \chi_V = \chi_{VUV} = \chi_U$. \square

Examples of groupoid algebras of ample groupoids include group algebras, Leavitt path algebras [6, 7] and inverse semigroup algebras [25], as well as discrete groupoid algebras and certain cross product and partial action cross product algebras. In general, groupoid algebras allow one to construct discrete analogues of a number of classical C^* -algebras that can be realized as C^* -algebras of ample groupoids [9, 20, 22].

3. THE EQUIVALENCE THEOREM

Fix an ample groupoid \mathcal{G} and a commutative ring with unit \mathbb{k} . Our goal is to establish an equivalence between the category $\text{mod-}\mathbb{k}\mathcal{G}$ of unitary right $\mathbb{k}\mathcal{G}$ -modules and the category $\mathcal{B}_{\mathbb{k}}\mathcal{G}$ of \mathcal{G} -sheaves of \mathbb{k} -modules. Let us recall the missing definitions.

If R is a ring with local units, a right R -module M is *unitary* if $MR = M$, or equivalently, for each $m \in M$, there is an idempotent $e \in E(R)$ with $me = m$. We write $\text{mod-}R$ for the category of unitary right R -modules. Two rings R, S with local units are *Morita equivalent* if $\text{mod-}R$ is equivalent to $\text{mod-}S$ [1, 3, 10]. One can equivalently define Morita equivalence in terms of unitary left modules and in terms of Morita contexts [1, 3, 10].

Suppose that R is a \mathbb{k} -algebra with local units. Then we note that every unitary R -module is a \mathbb{k} -module and the \mathbb{k} -module structure is compatible with the \mathbb{k} -algebra structure. Indeed, if $e \in E(R)$, then Me is a unital eMe -module and hence a \mathbb{k} -module in the usual way. As M is unitary, it follows that M is the directed union $\bigcup_{e \in E(M)} Me$ and hence a \mathbb{k} -module. More concretely, the \mathbb{k} -module structure is given as follows: if $c \in \mathbb{k}$ and $m \in M$, then $cm = m(ce)$ where e is any idempotent such that $me = m$. The \mathbb{k} -module structure is then automatically preserved by any R -module homomorphism, as in the case of unital rings.

Define a functor $\Gamma_c: \mathcal{B}_{\mathbb{k}}\mathcal{G} \rightarrow \text{mod-}\mathbb{k}\mathcal{G}$ as follows. If (E, p) is a \mathcal{G} -sheaf of \mathbb{k} -modules, then $\Gamma_c(E, p)$ is the set of all compactly supported (global) sections $s: \mathcal{G}^{(0)} \rightarrow E$ of p with pointwise addition. We define a $\mathbb{k}\mathcal{G}$ -module structure by

$$(sf)(x) = \sum_{\mathbf{d}(g)=x} f(g)s(\mathbf{r}(g))g = \sum_{\mathbf{d}(g)=x} f(g)R_g(s(\mathbf{r}(g))).$$

As usual, the sum is finite because f is a finite sum of functions with compact support and the fibers of \mathbf{d} are closed and discrete. It is easy to check that this makes $\Gamma_c(E, p)$ into a $\mathbb{k}\mathcal{G}$ -module and that the induced \mathbb{k} -module structure is just the pointwise one. The following observation is so fundamental that we shall often use it without comment throughout.

Proposition 3.1. *If $U \in \text{Bis}_c(\mathcal{G})$ and $s \in \Gamma_c(E, p)$, then*

$$(s\chi_U)(x) = \begin{cases} s(\mathbf{r}(g))g, & \text{if } g \in U, \mathbf{d}(g) = x \\ 0, & \text{if } x \notin U^{-1}U. \end{cases}$$

In particular, if $U \subseteq \mathcal{G}^{(0)}$ is compact open, then $(s\chi_U) = \chi_U(x)s(x)$.

The module $\Gamma_c(E, p)$ is unitary because if $s: \mathcal{G}^{(0)} \rightarrow E$ has compact support, then we can find a compact open set U containing the support of s (just cover the support by compact open sets and take the union of a finite subcover). Then one readily checks that $s\chi_U = s$ using Proposition 3.1.

If

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & \searrow p \quad \swarrow q & \\ & \mathcal{G}^{(0)} & \end{array}$$

is a morphism of \mathcal{G} -sheaves of \mathbb{k} -modules and $s \in \Gamma_c(E, p)$, then define $\Gamma_c(\varphi)(s) = \varphi \circ s$. It is straightforward to verify that Γ_c is a functor.

Conversely, let M be a unitary right $\mathbb{k}\mathcal{G}$ -module. We define a \mathcal{G} -sheaf $\text{Sh}(M) = (\widetilde{M}, p_M)$ in steps. Recall that we have been using \mathcal{B} to denote the generalized boolean algebra of compact open subsets of $\mathcal{G}^{(0)}$. For each $U \in \mathcal{B}$, we can consider the \mathbb{k} -submodule $M(U) = M\chi_U$. If $U \subseteq V$, then $M(U) = M\chi_U = M\chi_{UV} = M\chi_U\chi_V \subseteq M\chi_V = M(V)$. Note that $M(U)$ is a $\mathbb{k}\mathcal{G}_U = \chi_U \cdot \mathcal{G} \cdot \chi_U$ -module and since $\mathbb{k}\mathcal{G} = \bigcup_{U \in \mathcal{B}} \chi_U \cdot \mathcal{G} \cdot \chi_U = \varinjlim_{U \in \mathcal{B}} \mathbb{k}\mathcal{G}_U$, it follows that $M = \bigcup_{U \in \mathcal{B}} M(U) = \varinjlim_{U \in \mathcal{B}} M(U)$, where the latter has the obvious module structure coming from $\mathbb{k}\mathcal{G} = \varinjlim_{U \in \mathcal{B}} \mathbb{k}\mathcal{G}_U$.

Let $x \in \mathcal{G}^{(0)}$. If $x \in V \subseteq U$ with $U, V \in \mathcal{B}$, then we have a \mathbb{k} -module homomorphism $\rho_V^U: M(U) \rightarrow M(V)$ given by $m \mapsto m\chi_V$. Since $\rho_U^U = 1_{M(U)}$ and if $W \subseteq V \subseteq U$, we have $\rho_W^V \circ \rho_V^U = \rho_W^U$, it follows that we can form the direct limit \mathbb{k} -module $M_x = \varinjlim_{x \in U} M(U)$. If $m \in M(U)$, we let $[m]_x$ denote the equivalence class of m in M_x . Since $M = \bigcup_{U \in \mathcal{B}} M(U)$ and each element of \mathcal{B} is contained in an element which contains x , it follows that $[m]_x$ is defined for all $m \in M$ and $m \mapsto [m]_x$ gives a \mathbb{k} -linear map $M \rightarrow M_x$.

Put $\widetilde{M} = \coprod_{x \in \mathcal{G}^{(0)}} M_x$ and let $p_M(M_x) = x$ for $x \in \mathcal{G}^{(0)}$. Let U be a compact open subset of $\mathcal{G}^{(0)}$ and let $m \in M$. Define

$$(U, m) = \{[m]_x \mid x \in U\} \subseteq \widetilde{M}.$$

Suppose that $[m]_x \in (U, m_1) \cap (V, m_2)$. Then there is a compact open neighborhood $W \subseteq U \cap V$ of x such that $m\chi_W = m_1\chi_W = m_2\chi_W$. It follows that $[m]_x \in (W, m) \subseteq (U, m_1) \cap (V, m_2)$ and hence the sets (U, m) form a basis for a topology on \widetilde{M} . Continuity of p_M follows because if U is a compact open subset of $\mathcal{G}^{(0)}$, then $p_M^{-1}(U) = \bigcup_{m \in M} (U, m)$ is open. Trivially, p_M takes (U, m) bijectively to U and is thus a local homeomorphism.

Each stalk M_x is a \mathbb{k} -module. We must show continuity of the \mathbb{k} -module structure. To establish continuity of the zero section $x \mapsto [0]_x$, suppose that (U, m) is a basic neighborhood of $[0]_x$. Then there is a compact open neighborhood W of x with $W \subseteq U$ and $m\chi_W = 0$. Then, for all $z \in W$, one has $[m]_z = [0]_z$ and so the zero section maps W into (U, m) . Thus the zero section is continuous.

To see that scalar multiplication is continuous, let $k \in \mathbb{k}$ and suppose that $[kn]_x = k[n]_x \in (U, m)$. Then there is a compact open neighborhood W of x with $W \subseteq U$ and $kn\chi_W = m\chi_W$. If $(k, [n]_z) \in \{k\} \times (W, n)$, then $k[n]_z = [kn]_z = [m]_z$ because $z \in W$ and $kn\chi_W = m\chi_W$. This yields continuity of scalar multiplication.

Continuity of addition is proved as follows. Suppose (U, m) is a basic neighborhood of $[m_1]_x + [m_2]_x = [m_1 + m_2]_x$. Then there is a compact open neighborhood W of x with $W \subseteq U$ and $(m_1 + m_2)\chi_W = m\chi_W$. Therefore, if $([m_1]_z, [m_2]_z) \in ((m_1, W) \times (m_2, W)) \cap (\widetilde{M} \times_{\mathcal{G}(0)} \widetilde{M})$, then $[m_1]_z + [m_2]_z = [m_1 + m_2]_z = [m]_z \in (U, m)$. Therefore, addition is continuous.

Next, we must define the \mathcal{G} -action. Define, for $g \in \mathcal{G}^{(1)}$, a mapping $R_g: M_{\mathbf{r}(g)} \rightarrow M_{\mathbf{d}(g)}$ by $R_g([m]_{\mathbf{r}(g)}) = [m\chi_U]_{\mathbf{d}(g)}$ where U is a compact local bisection containing g . We also write $R_g([m]_{\mathbf{r}(g)}) = [m]_{\mathbf{r}(g)}g$.

Proposition 3.2. *The following hold.*

- (1) R_g is a well-defined \mathbb{k} -module homomorphism.
- (2) If $(g, h) \in \mathcal{G}^{(2)}$, then $([m]_{\mathbf{r}(g)}g)h = [m]_{\mathbf{r}(gh)}(gh)$.
- (3) If $x \in \mathcal{G}^{(0)}$, then $[m]_x x = [m]_x$.

Proof. Suppose that $g: y \rightarrow x$. To show that R_g is well defined, let $[m]_x = [n]_x$ and let $U, V \in \text{Bis}_c(\mathcal{G})$ with $g \in U \cap V$. Then there exist a compact open neighborhood W of x with $m\chi_W = n\chi_W$ and $Z \in \text{Bis}_c(\mathcal{G})$ such that $g \in Z \subseteq U \cap V$. Note that $g \in WZ \subseteq U \cap V$ and so $y \in Z^{-1}WZ \subseteq \mathcal{G}^{(0)}$. Also we compute

$$\begin{aligned} m\chi_U\chi_{Z^{-1}WZ} &= m\chi_{UZ^{-1}WZ} = m\chi_{U(WZ)^{-1}WZ} = m\chi_{WZ} = m\chi_W\chi_Z \\ &= n\chi_W\chi_Z = n\chi_{WZ} = n\chi_{V(WZ)^{-1}WZ} = n\chi_{VZ^{-1}WZ} \\ &= n\chi_V\chi_{Z^{-1}WZ} \end{aligned}$$

which shows that $[m\chi_U]_y = [n\chi_V]_y$, i.e., R_g is well defined. Clearly R_g is \mathbb{k} -linear.

Suppose now that $(g, h) \in \mathcal{G}^{(2)}$. Choose $U, V \in \text{Bis}_c(\mathcal{G})$ such that $g \in U$ and $h \in V$. Then $gh \in UV$ and so if $g: y \rightarrow x$ and $h: z \rightarrow y$, then

$$([m]_x g)h = [m\chi_U]_y h = [m\chi_U\chi_V]_z = [m\chi_{UV}]_z = [m]_z(gh)$$

as required.

Finally, if $x \in \mathcal{G}^{(0)}$ and U is a compact open neighborhood of x in $\mathcal{G}^{(0)}$, then $[m]_x x = [m\chi_U]_x = [m]_x$ by definition of M_x . \square

In light of Proposition 3.2, there is an action map $\widetilde{M} \times_{\mathcal{G}(0)} \mathcal{G}^{(1)} \rightarrow \widetilde{M}$ given by $([m]_{\mathbf{r}(g)}, g) \mapsto [m]_{\mathbf{r}(g)}g$ satisfying $p_M([m]_{\mathbf{r}(g)}g) = \mathbf{d}(g)$. To prove that $\text{Sh}(M)$ is a \mathcal{G} -sheaf of \mathbb{k} -modules, it remains to show the action map is continuous. Let $g: y \rightarrow x$ and let (m, U) be a basic neighborhood of $[n]_x g$. Then $y \in U$ and $[m]_y = [n]_x g$. Let $V \in \text{Bis}_c(\mathcal{G})$ with $g \in V$. Then $[n\chi_V]_y = [m]_y$ and so there exists a compact open neighborhood $W \subseteq U$ of y with $n\chi_{VW} = n\chi_V\chi_W = m\chi_W$. Note that $g \in VW$ and $x \in VWV^{-1} \subseteq \mathcal{G}^{(0)}$.

Consider the neighborhood $N = ((n, VWV^{-1}) \times VW) \cap (\widetilde{M} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)})$ of $([n]_x, g)$. If $([n]_z, h) \in N$, with $h: z' \rightarrow z$, then because $h \in VW$, we have $[n]_z h = [n\chi_{VW}]_{z'} = [m\chi_W]_{z'} = [m]_{z'} \in (m, U)$ as $z' \in V^{-1}VW \subseteq W \subseteq U$. This establishes that (\widetilde{M}, p) is a \mathcal{G} -sheaf of \mathbb{k} -modules.

Next suppose that $f: M \rightarrow N$ is a $\mathbb{k}\mathcal{G}$ -module homomorphism. The $f(M(U)) = f(M\chi_U) = f(M)\chi_U \subseteq N\chi_U = N(U)$. Thus there is an induced \mathbb{k} -linear map $f_x: M_x \rightarrow N_x$ given by $f([m]_x) = [f(m)]_x$. Define

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\text{Sh}(f)} & \widetilde{N} \\ & \searrow p_M \quad \swarrow p_N & \\ & \mathcal{G}^{(0)} & \end{array}$$

by $\text{Sh}(f)([m]_x) = f_x([m]_x)$. First we check that $\text{Sh}(f)$ preserves the action. Suppose $g: y \rightarrow x$ and $U \in \text{Bis}_c(\mathcal{G})$ with $g \in U$. Then

$$f_y([m]_x g) = f_y([m\chi_U]_y) = [f(m\chi_U)]_y = [f(m)\chi_U]_y = [f(m)]_x g = f_x([m]_x)g$$

as required.

It remains to check continuity of $\text{Sh}(f)$. Let $[m]_x \in \widetilde{M}$ and let (U, n) be a basic neighborhood of $f_x([m]_x)$. Then $x \in U$ and $f_x([m]_x) = [f(m)]_x = [n]_x$. Choose a compact open neighborhood W of x contained in U such that $f(m)\chi_W = n\chi_W$. Consider the neighborhood (W, m) of $[m]_x$. If $[m]_z \in (W, m)$, then $f_z([m]_z) = [f(m)]_z = [f(m)\chi_W]_z = [n\chi_W]_z = [n]_z$ because $z \in W$. Thus $\text{Sh}(f)(W, m) \subseteq (U, n)$, yielding the continuity of $\text{Sh}(f)$. It is obvious that Sh is a functor.

The following lemma will be useful for proving that these functors are quasi-inverse.

Lemma 3.3. *Let $M \in \text{mod-}\mathbb{k}\mathcal{G}$. If $U \in \text{Bis}_c(\mathcal{G})$ and $x \notin U^{-1}U$, then $[m\chi_U]_x = 0$.*

Proof. Since $U^{-1}U$ is compact and $\mathcal{G}^{(0)}$ is Hausdorff, we can find a compact open neighborhood W of x with $W \cap U^{-1}U = \emptyset$. Then $m\chi_U\chi_W = m\chi_{UW} = 0$. \square

Theorem 3.4. *There are natural isomorphisms $\Gamma_c \circ \text{Sh} \cong 1_{\text{mod-}\mathbb{k}G}$ and $\text{Sh} \circ \Gamma_c \cong 1_{\mathcal{B}_{\mathbb{k}\mathcal{G}}}$. Hence the categories $\text{mod-}\mathbb{k}\mathcal{G}$ and $\mathcal{B}_{\mathbb{k}\mathcal{G}}$ are equivalent.*

Proof. Let M be a unitary $\mathbb{k}\mathcal{G}$ -module and define $\eta_M: M \rightarrow \Gamma_c(\text{Sh}(M))$ by $\eta_M(m) = s_m$ where $s_m(x) = [m]_x$ for all $x \in X$. We claim that s_m is continuous with compact support. Continuity is easy: if $s_m(x) \in (U, n)$, then $x \in U$ and $[m]_x = [n]_x$. So there is a compact open neighborhood W of x with $W \subseteq U$ and $m\chi_W = n\chi_W$. Then if $z \in W$, we have $s_m(z) = [m]_z = [m\chi_W]_z = [n\chi_W]_z = [n]_z \in (U, n)$. Thus s_m is continuous. We claim that the support of s_m is compact. Let $U \in \mathcal{B}$ with $m\chi_U = m$. Suppose that $x \notin U$. Then Lemma 3.3 implies that $[m]_x = [m\chi_U]_x = 0$. Thus the support of s_m is a closed subset of U and hence compact.

We claim that η_M is an isomorphism (it is clearly natural in M). Let us first show that η_M is a module homomorphism. It is clearly \mathbb{k} -linear and hence it suffices to show that if $U \in \text{Bis}_c(\mathcal{G})$, then $\eta_M(m\chi_U) = \eta_M(m)\chi_U$. Note that $\eta_M(m\chi_U) = s_{m\chi_U}$. If $x \notin U^{-1}U$, then $s_{m\chi_U}(x) = [m\chi_U]_x = 0$ by Lemma 3.3. If $x = \mathbf{d}(g)$ with $g \in U$, then we have $s_{m\chi_U}(x) = [m\chi_U]_x = [m]_{\mathbf{r}(g)}g = s_m(\mathbf{r}(g))g$. Therefore, in light of Proposition 3.1, we conclude that $s_{m\chi_U} = s_m\chi_U$. This shows that η_M is a $\mathbb{k}\mathcal{G}$ -module homomorphism.

Suppose that $0 \neq m \in M$. Then $m\chi_U = m$ for some $U \in \mathcal{B}$. Let \mathcal{B}_U be the boolean ring of compact open subsets of U . Let I be the ideal of \mathcal{B}_U consisting of those V with $m\chi_V = 0$. This is a proper ideal (since $U \notin I$) and hence contained in a maximal ideal \mathfrak{m} . Let x be the point of U corresponding to \mathfrak{m} under Stone duality. Then the ultrafilter of compact open neighborhoods of x is $\mathcal{B} \setminus \mathfrak{m}$ and hence does not intersect I . Therefore, $m\chi_V \neq 0$ for all compact open neighborhoods of x , that is, $s_m(x) = [m]_x \neq 0$. Therefore, $\eta_M(m) = s_m \neq 0$ and so η_M is injective.

To see that η_M is surjective, let $s \in \Gamma_c(\text{Sh}(M))$ and let K be the support of s . For each $x \in K$, we can find a compact open neighborhood U_x of x and an element $m_x \in M$ such that $s(z) = [m_x]_z$ for all $z \in U_x$ (choose U_x mapping under s into a basic neighborhood of $s(x)$ of the form (V_x, m_x)). By compactness of K , we can find a finite subcover of the U_x with $x \in X$. Since $\mathcal{G}^{(0)}$ is Hausdorff, we can refine the subcover by a partition into compact open subsets, that is, we can find disjoint compact open sets V_1, \dots, V_n and elements $m_1, \dots, m_n \in M$ such that $K \subseteq V_1 \cup \dots \cup V_n$ and $s(x) = [m_i]_x$ for all $x \in V_i$. Consider $m = m_1\chi_{V_1} + \dots + m_n\chi_{V_n}$. Then $m\chi_{V_i} = m_i\chi_{V_i}$ and so $[m]_x = [m_i]_x = s(x)$ for all $x \in V_i$. We conclude that $[m]_x = s(x)$ for all $x \in V_1 \cup \dots \cup V_n$. If $x \notin V_1 \cup \dots \cup V_n$, then $x \notin K$ and so $s(x) = 0$. But also $[m]_x = \sum_{i=1}^n [m_i\chi_{V_i}]_x = 0$ by Lemma 3.3. Thus $s(x) = [m]_x = s_m(x)$ for all $x \in \mathcal{G}^{(0)}$ and hence $s = \eta_M(m)$. This concludes the proof η_M is an isomorphism.

Next let (E, p) be a \mathcal{G} -sheaf of \mathbb{k} -modules and put $M = \Gamma_c(E, p)$. We define an isomorphism

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\varepsilon_{(E,p)}} & E \\ & \searrow p_M \quad \swarrow p & \\ & \mathcal{G}^{(0)} & \end{array}$$

of $\text{Sh}(\Gamma_c(E, p))$ and (E, p) as follows. Define $\varepsilon_{(E,p)}([s]_x) = s(x)$ for $s \in M$ and $x \in \mathcal{G}^{(0)}$. This is well defined because if $s\chi_U = s'\chi_U$ for some compact open neighborhood U of x , then $s(x) = s'(x)$ by Proposition 3.1. Also, $p(s(x)) = x = p_M([s]_x)$, whence $p \circ \varepsilon_{(E,p)} = p_M$. Clearly, $\varepsilon_{(E,p)}$ restricts to a \mathbb{k} -module homomorphism on each fiber. Let $g: y \rightarrow x$ and suppose $U \in \text{Bis}_c(\mathcal{G})$ with $g \in U$, then $\varepsilon_{(E,p)}([s]_x g) = \varepsilon_{(E,p)}([s\chi_U]_y) = (s\chi_U)(y) = s(x)g = \varepsilon_{(E,p)}([s]_x)g$ (using Proposition 3.1). It therefore remains to prove that $\varepsilon_{(E,p)}$ is a homeomorphism.

To see that $\varepsilon_{(E,p)}$ is continuous, let $[s]_x \in \widetilde{M}$ and let U be a neighborhood of $\varepsilon_{(E,p)}([s]_x) = s(x)$. Let W be a compact open neighborhood of x with $s(W) \subseteq U$. Consider the neighborhood (W, s) of $[s]_x$. Then, for $[s]_z \in (W, s)$, we have $\varepsilon_{(E,p)}([s]_z) = s(z) \in U$. Thus $\varepsilon_{(E,p)}$ is continuous. As p, p_M are local homeomorphisms, we deduce from $p \circ \varepsilon_{(E,p)} = p_M$ that $\varepsilon_{(E,p)}$ is a local homeomorphism and hence open. It remains to prove that $\varepsilon_{(E,p)}$ is bijective.

Suppose that $s(x) = \varepsilon_{(E,p)}([s]_x) = \varepsilon_{(E,p)}([t]_x) = t(x)$. Choose a neighborhood U of $s(x) = t(x)$ such that $p|_U$ is a homeomorphism onto its image. Let W be a compact open neighborhood of x such that both $s(W) \subseteq U$ and $t(W) \subseteq U$. Then if $z \in W$, we have $p(s(z)) = z = p(t(z))$ and $s(z), t(z) \in U$ and hence $s(z) = t(z)$. Thus $s\chi_W = t\chi_W$ (cf. Proposition 3.1) and so $[s]_x = [t]_x$. This yields injectivity of $\varepsilon_{(E,p)}$. Next let $e \in E_x$. Let U be neighborhood of e such that $p|_U: U \rightarrow p(U)$ is a homeomorphism. Let W be a compact open neighborhood of x contained in $p(U)$ and define $s \in \Gamma_c(E, p)$ to agree with $(p|_U)^{-1}$ on W and be 0 outside of W . Then $\varepsilon_{(E,p)}([s]_x) = s(x) = e$. Clearly, $\varepsilon_{(E,p)}$ is natural (E, p) . This completes the proof. \square

As a corollary, we recover the main result of [7], and moreover we do not require the Hausdorff assumption.

Corollary 3.5. *Let \mathcal{G} and \mathcal{H} be Morita equivalent ample groupoids. Then $\mathbb{k}\mathcal{G}$ is Morita equivalent to $\mathbb{k}\mathcal{H}$ for any commutative ring with unit \mathbb{k} .*

By restricting to the case where $\mathcal{G}^{(1)} = \mathcal{G}^{(0)}$, we also have the following folklore result.

Corollary 3.6. *Let X be a Hausdorff space with a basis of compact open subsets and \mathbb{k} a commutative ring with unit. Let $C_c(X, \mathbb{k})$ be the ring of locally constant functions $X \rightarrow \mathbb{k}$ with compact support. Then the category of sheaves of \mathbb{k} -modules on X is equivalent to the category of unitary $C_c(X, \mathbb{k})$ -modules.*

If \mathcal{G} is a discrete groupoid, then $\mathcal{B}\mathcal{G}$ is equivalent to the category $\mathbf{Set}^{\mathcal{G}^{op}}$ of contravariant functors from \mathcal{G} to the category of sets [11, 12]. Therefore, $\mathcal{B}_{\mathbb{k}}\mathcal{G}$ is equivalent to the category $(\text{mod-}\mathbb{k})^{\mathcal{G}^{op}}$ of contravariant functors from \mathcal{G} to $\text{mod-}\mathbb{k}$. It is well known that $(\text{mod-}\mathbb{k})^{\mathcal{G}^{op}}$ is equivalent to $\text{mod-}\mathbb{k}\mathcal{G}$ when $\mathcal{G}^{(0)}$ is finite [15] and presumably the following extension is also well known, although the author doesn't know a reference.

Corollary 3.7. *Let \mathcal{G} be a discrete groupoid and \mathbb{k} a commutative ring with unit. Then $\text{mod-}\mathbb{k}\mathcal{G}$ is equivalent to the category $(\text{mod-}\mathbb{k})^{\mathcal{G}^{op}}$ of contravariant functors $\mathcal{G} \rightarrow \text{mod-}\mathbb{k}$. Hence naturally equivalent discrete groupoids have Morita equivalent algebras.*

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